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Plane symmetric vacuum solutions in the Brans–Dicke theory of gravitation

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Abstract. A class of exact plane symmetric solutions of the Brans–Dicke vacuum field equations is obtained. This class, in the limit $\omega \to \infty$, is found to agree with Taub's class of solutions. It has been observed that the non-static solution belonging to this class represents a plane-fronted wave.

1. Introduction

Brans and Dicke (1961) have proposed a theory of gravitation (known as the Brans–Dicke, BD, theory) by introducing a long-range scalar field to modify Einstein's theory in such a way as to make it more compatible with the requirements of Mach's principle. It is well known that this theory can have two alternative mathematical representations. In one representation of the theory, usually known as the canonical representation, the field equations are

$$R_{ij}-\tfrac{1}{2}Rg_{ij}=-\frac{8\pi}{\phi}T_{ij}-\frac{\omega}{\phi^2}(\phi_{,i}\phi_{,j}-\tfrac{1}{2}g_{ij}\phi_{,k}\phi^{,k})-\frac{1}{\phi}(\phi_{;ij}-g_{ij}\phi_{;k}^{;k}) \tag{1}$$

and

$$\phi_{;a}^{;a}=\frac{8\pi}{3+2\omega}T \tag{2}$$

where the parameter ω is a dimensionless constant. Indices following a comma or semicolon denote partial or covariant differentiations, respectively. The other representation of the theory, known as Dicke's representation (Dicke 1962), is obtained from the canonical representation by a unit transformation, where the units of length, time and reciprocal mass are changed by a scale factor, so that

$$g_{ij}=\lambda^{-1}\bar{g}_{ij}, \qquad m=\lambda^{1/2}\bar{m}, \qquad \phi=\lambda\bar{\phi} \tag{3}$$

where λ is a function of ϕ and $\bar{\phi}$ is taken to be the constant G_0 (the Newtonian gravitational constant). In this representation the field equations are

$$\bar{R}_{ij}-\tfrac{1}{2}\bar{R}\bar{g}_{ij}=-8\pi G_0\bar{T}_{ij}-\tfrac{1}{2}(2\omega+3)(\Lambda_{,i}\Lambda_{,j}-\tfrac{1}{2}\bar{g}_{ij}\Lambda_{,k}\Lambda^{,k}) \tag{4}$$

and

$$\bar{g}^{ab}\Lambda_{:ab}=\frac{8\pi G_0}{2\omega+3}\bar{T} \tag{5}$$

where $\Lambda \equiv \ln(\phi/\bar{\phi})$ and a colon followed by an index represents a covariant differentiation with respect to $\bar{g}_{ab}$. Though physically equivalent, these forms have their individual significance in view of convenience in mathematical calculations. As a matter of fact, one always finds the canonical representation to be convenient in the study of the equations of motion in the BD theory, whereas Dicke's representation is convenient in the discussion of the phenomena of scalar waves (Dicke 1964). However, the field equations in both cases agree with Einstein's field equations when $\omega \to \infty$ and $\phi \to G_0^{-1}$.

In pursuing the study of the BD theory it has been common practice to compare the results of this theory with those of Einstein's theory under similar conditions, to find out the influence of the ϕ-field, if any. With this objective, the problem of finding an exact plane symmetric solution to the BD vacuum field equations has been taken up. It is well known in Einstein's theory that no non-static vacuum solution exists exhibiting either spherical symmetry (Birkhoff's theorem) or plane symmetry (Taub 1951). However, it was observed by Reddy (1973) that only for a static scalar field is the Birkhoff theorem true in the BD theory. This suggests that the time dependence of the scalar field and the metric tensor go together in this case. So far, however, no such exact solution has been obtained. In the present paper, while investigating the plane symmetric BD vacuum fields, a class of exact solutions has been obtained which contains a non-static solution. The method of obtaining the solutions, as illustrated in § 3, is such that it naturally leads us to discuss the cases discussed by Taub (1951). Hence, comparison between our class of solutions with those of Taub is conveniently made. Moreover, the existence of a non-static solution shows a possibility of realizing plane gravitational waves in the BD theory. To study this aspect of the solution, we have transformed it by using the unit transformation (3) so that the transformed solution satisfies the field equations (4) and (5) in Dicke's representation. It is observed that this solution represents an expansion-free radiation field of Petrov type O, and hence, a plane-fronted wave (Kundt 1961). It should, however, be mentioned that the energy transport by such waves can be studied in view of the conservation laws in the BD theory (Nutku 1969, Lee 1974). In fact, this part of the investigation is still under progress and will shortly be communicated.

2. The field equations

The plane symmetric metric of Taub (1951) is given as

$$(ds)^2 = e^{2u}(dt^2 - dx^2) - e^{2v}(dy^2 + dz^2) \tag{6}$$

where u and v are functions of x and t only. The BD field equations (1) and (2) with respect to the metric (6) are

$$R_{11} \equiv u_{,11} - u_{,44} + 2(v_{,11} + v_{,1}^2 - v_{,4}u_{,4} - v_{,1}u_{,1}) = -\frac{\omega}{\phi^2}\phi_{,1}^2 - \frac{1}{\phi}(\phi_{,11} - u_{,1}\phi_{,1} - u_{,4}\phi_{,4}) \tag{7}$$

$$R_{22} = R_{33} \equiv v_{,44} - v_{,11} + 2(v_{,4}^2 - v_{,1}^2) = \frac{1}{\phi}(v_{,1}\phi_{,1} - v_{,4}\phi_{,4}) \tag{8}$$

$$R_{44} \equiv u_{,44} - u_{,11} + 2(v_{,44} + v_{,4}^2 - v_{,4}u_{,4} - v_{,1}u_{,1}) = -\frac{\omega}{\phi^2}\phi_{,4}^2 - \frac{1}{\phi}(\phi_{,44} - u_{,1}\phi_{,1} - u_{,4}\phi_{,4}) \tag{9}$$

$$R_{14} \equiv 2(v_{,14} - v_{,1}v_{,4} - v_{,4}u_{,1} - v_{,1}u_{,4}) = -\frac{\omega}{\phi^2}\phi_{,1}\phi_{,4} - \frac{1}{\phi}(\phi_{,14} - u_{,4}\phi_{,1} - u_{,1}\phi_{,4}) \tag{10}$$

and

$$\phi_{,44}-\phi_{,11}-2v_{,1}\phi_{,1}+2v_{,4}\phi_{,4}=0 \tag{11}$$

where indices 4 and 1 after a comma denote partial differentiation with respect to t and x respectively. In our analysis we refer to the set of equations (7), (8), (9) and (10) as (A).

3. Solutions of the field equations

On multiplying equation (11) by e^{2v} we get

$$\frac{\partial}{\partial t}(\phi_{,4}\,e^{2v})-\frac{\partial}{\partial x}(\phi_{,1}\,e^{2v})=0. \tag{12}$$

Equation (8), on multiplying by $\phi\,e^{2v}$ on both sides, reduces to

$$\frac{\partial}{\partial t}\left(\phi\frac{\partial}{\partial t}\,e^{2v}\right)-\frac{\partial}{\partial x}\left(\phi\frac{\partial}{\partial x}\,e^{2v}\right)=0. \tag{13}$$

From (12) and (13) on adding, we get

$$\frac{\partial^2}{\partial t^2}(\phi\,e^{2v})-\frac{\partial^2}{\partial x^2}(\phi\,e^{2v})=0 \tag{14}$$

which has a plane-wave solution:

$$\phi\,e^{2v}=f(x+t)+g(x-t) \tag{15}$$

where f and g are arbitrary functions of the variables indicated.

Now let us consider another plane symmetric metric

$$(ds)^2=e^{2u}(dt^2-dx^2)-e^{2\chi}(dy^2+dz^2) \tag{16}$$

where u and χ are functions of x and t. The component $R_{22}=R_{33}=0$ of Einstein's vacuum field equations corresponding to the metric (16) (we refer to this set of equations as B) suggests that $e^{2\chi}$ behaves like a plane wave (Taub 1951). This leads us to identify $\phi\,e^{2v}$ in (15) with $e^{2\chi}$, so that

$$\phi\,e^{2v}=e^{2\chi}=f(x+t)+g(x-t). \tag{17}$$

By defining an auxiliary function w as

$$\phi=e^{w} \tag{18}$$

we get from (17)

$$2\chi=w+2v=\ln(f+g). \tag{19}$$

Now, eliminating χ from the set of equations (B) in terms of w and v in view of (19), we arrive at the following set of equations (to be referred to as C):

$$R_{11} \equiv u_{,11} - u_{,44} + 2(v_{,11} + v_{,1}^2 - v_{,4}u_{,4} - v_{,1}u_{,1})$$
$$= -w_{,11} - \tfrac{1}{2}w_{,1}^2 - 2w_{,1}v_{,1} + w_{,4}u_{,4} + w_{,1}u_{,1} \tag{20}$$
$$R_{22} = R_{33} \equiv v_{,44} - v_{,11} + 2(v_{,4}^2 - v_{,1}^2)$$
$$= -\tfrac{1}{2}(w_{,44} - w_{,11}) - \tfrac{1}{2}(w_{,4}^2 - w_{,1}^2) - 2(w_{,4}v_{,4} - w_{,1}v_{,1}) \tag{21}$$
$$R_{44} \equiv u_{,44} - u_{,11} + 2(v_{,44} + v_{,4}^2 - v_{,1}u_{,1} - v_{,4}u_{,4})$$
$$= -w_{,44} - \tfrac{1}{2}w_{,4}^2 - 2w_{,4}v_{,4} + w_{,1}u_{,1} + w_{,4}u_{,4} \tag{22}$$

and

$$R_{14} \equiv 2(v_{,14} + v_{,1}v_{,4} - v_{,4}u_{,1} - v_{,1}u_{,4})$$
$$= -w_{,14} - \tfrac{1}{2}w_{,1}w_{,4} - w_{,1}v_{,4} - v_{,1}w_{,4} + w_{,4}u_{,1} + w_{,1}u_{,4}. \tag{23}$$

This set of equations (C) is identical to the set of equations (A) if the following set of relationships, (to be referred to as D), is true:

$$(w_{,44} - w_{,11}) + [w_{,4}^2 - w_{,1}^2 + 2(v_{,4}w_{,4} - v_{,1}w_{,1})] = 0 \tag{24}$$
$$(\omega + \tfrac{1}{2})w_{,4}^2 = 2v_{,4}w_{,4} \tag{25}$$
$$(\omega + \tfrac{1}{2})w_{,1}^2 = 2v_{,1}w_{,1} \tag{26}$$

and

$$(\omega + \tfrac{1}{2})w_{,1}w_{,4} = w_{,1}v_{,4} + w_{,4}v_{,1}. \tag{27}$$

It can easily be seen that (24) is the wave equation (11). This leads to the conclusion that the set of equations (A) with (11) is fully identical to the set of equations (C) with (24) when equations (25), (26) and (27) are satisfied. But equations (25), (26) and (27) suggest that v is a function of w such that

$$v = \tfrac{1}{4}(2\omega + 1)w.$$

From (19) we obtain

$$w = \frac{2}{2\omega + 3}\ln(f + g) \tag{28}$$

and therefore

$$v = \frac{2\omega + 1}{2(2\omega + 3)}\ln(f + g). \tag{29}$$

Thus the u solution of equations (B) along with (28) and (29) will constitute the solution of equations (A) and (11). That is, the solution of the BD field equations (we call it S) is

$$u = -\tfrac{1}{4}\ln(f + g) + \eta(x + t) + \kappa(x - t) \tag{30}$$
$$\phi = e^w = (f + g)^{2/(2w+3)} \tag{31}$$

and

$$v = \frac{2\omega + 1}{2(2\omega + 3)}\ln(f + g) \tag{32}$$

where η and κ are arbitrary functions of the variables indicated.

Since the set of equations (B) is equivalent to (A) through (C) and under the condition (D), the relations between f, g, η and κ are the same as obtained by Taub (1951) and thus lead to the same three cases as discussed by him (Taub 1951). We now discuss solution (S) i.e. equations (30), (31) and (32) under these cases in the same order as done by Taub (1951).

It can easily be verified that for case I, the values of ϕ and v from equations (27) and (28) become constant and the metric reduces to

$$(ds)^2 = e^{2(\eta+\kappa)}(dt^2 - dx^2) - (dy^2 + dz^2) \tag{33}$$

which can be transformed to a flat metric by using Taub's transformation for this case. For case II, the solution (S) from equations (30), (31) and (32) reduces to

$$u = -\tfrac{1}{4}\ln g + \eta + \tfrac{1}{2}\ln g' + c_1 \tag{34}$$

$$w = \frac{2}{2\omega+3}\ln g \tag{35}$$

and

$$v = \frac{2\omega+1}{2(2\omega+3)}\ln g. \tag{36}$$

Thus the metric is

$$(ds)^2 = e^{2\eta} g'/\sqrt{g}(dt^2 - dx^2) - (g)^{(2\omega+1)/(2\omega+3)}(dy^2 + dz^2). \tag{37}$$

Defining the transformations analogous to Taub (1951) as

$$1 + Y^1 + Y^4 = [g(x-t)]^{-(2\omega-1)/[2(2\omega+3)]}$$

and

$$Y^1 - Y^4 = -\frac{2(2\omega+3)}{2\omega-1}\int_0^{x+t} e^{2\eta(x+t)}\, d(x+t)$$

and applying these to equation (37) we get the transformed metric

$$(ds)^2 = (1 + Y^1 + Y^4)^{-2(2\omega+1)/(2\omega-1)}[(dY^4)^2 - (dY^1)^2 - (dY^2)^2 - (dY^3)^2] \tag{38}$$

which is conformally flat. From equation (35), the BD scalar ϕ is

$$\phi = e^w = (g)^{2/(2\omega+3)} = (1 + Y^1 + Y^4)^{-4/(2\omega-1)}. \tag{39}$$

For case III, the solution (S) can be transformed to a completely static solution by using the transformation used by Taub for this case. Then the metric (6) and the scalar ϕ are given, respectively, as

$$(ds)^2 = \frac{1}{\sqrt{(1+kX^1)}}[(dX^4)^2 - (dX^1)^2] - (1 + kX^1)^{(2\omega+1)/(2\omega+3)}[(dX^2)^2 + (dX^3)^2] \tag{40}$$

and

$$\phi = (1 + kX^1)^{2/(2\omega+3)}. \tag{41}$$

4. Conclusions

Here we have obtained a class of exact solutions which goes over to Taub's class of solutions in the limit when $\omega \to \infty$. It is interesting to note that our class of solutions contains a non-static solution, unlike its counterpart in Einstein's theory. Our discussion in this section will be confined to this non-static solution only.

The interesting feature of the solution is related to its wave character. This is revealed when we transform this solution (38) and (39) by making the unit transformation (3), so that it satisfies the vacuum field equation in Dicke's representation, namely,

$$\bar{R}_{ab} = -(\omega + \tfrac{3}{2})\Lambda_{,a}\Lambda_{,b} \tag{42}$$

and

$$\Lambda_{:a}^{:a} = 0. \tag{43}$$

It may be noted that the BD vacuum fields with null Λ will possess a congruence of null curves with tangent vector $l^a = \bar{g}^{ab}\Lambda_{,b}$ which will be necessarily geodetic (Pirani 1964), twist-free ($\frac{1}{2}l_{[a:b]}l^{a:b} = 0$) and expansion-free (from equation 43 such that $l^a_{:a} = 0$). In other words, the BD vacuum fields with null scalar Λ are equivalent to the fields with a normal (twist-free), non-expanding congruence of null geodesics with tangent vector $l^a = \bar{g}^{ab}\Lambda_{,b}$ satisfying the field equations

$$\bar{R}_{ab} = -(\omega + \tfrac{3}{2})l_{,a}l_{,b}.$$

This suggests that for $\omega \geqslant -\frac{3}{2}$ the BD vacuum fields with null scalar field Λ are expansion-free radiation fields (Kundt 1961). However, it should be mentioned here that the null congruence with such fields will be distortion-free also. The transformed form of our solution (38) and (39), by unit transformation (3), is characterized by the metric

$$(\mathrm{d}s)^2 = (1 + Y^1 + Y^4)^{-2(2\omega+3)/(2\omega-1)}[(\mathrm{d}Y^4)^2 - (\mathrm{d}Y^1)^2 - (\mathrm{d}Y^2)^2 - (\mathrm{d}Y^3)^2] \tag{44}$$

and the scalar

$$\phi = (1 + Y^1 + Y^4)^{-4/(2\omega-1)} \tag{45}$$

such that

$$\Lambda \equiv \ln\phi = -\frac{4}{(2\omega-1)}\ln(1 + Y^1 + Y^4).$$

This solution satisfies the field equations (42) and (43). It can be easily verified that Λ for our solution is a null field of the space time (44). Hence this solution represents an expansion-free radiation field. Moreover, the conformal flatness of the space time (44) suggests this field to be of Petrov type O. So, in view of the theorem due to Kundt (1961) that plane-fronted waves are non-expanding radiation fields of type N or O, it is concluded that our solution represents a plane-fronted wave.

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